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Use of the boundary element method to determine the thermal conductivity tensor of an anisotropic medium

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Abstract

In this paper, we propose a numerical algorithm to simultaneously predict the unknown conductivity coefficients and the unknown boundary data for a steady-state heat conduction problem in an anisotropic medium. The algorithm is based on a classical boundary element method (BEM) which is combined with a least squares technique. The numerical convergence and stability of the method proposed is investigated with respect to increasing the number of additional measurements provided and decreasing the amount of noise added into the input data. © 2001 Elsevier Science Ltd. All rights reserved.

1. Introduction

Anisotropic media occurs in nature, such as wood, crystals and sedimentary rocks, and can also be produced artificially, such as laminated and fiber-reinforced construction and electronic materials, cables, cylinders and tubes. Following the rapid increase of their industrial use, the understanding of heat conduction in this type of material is a great importance. In addition the production of oil and gas in many reservoirs is seriously affected by its highly anisotropic structure.

Experimentally, it is difficult to make accurate measurements of the thermal conductivity tensor, and analytically, it is difficult to solve the differential equation in which the elements of the tensor arise. Therefore numerical methods for anisotropic heat conduction problems appear to be a very useful tool. It is the purpose of this paper to develop a numerical algorithm to identify the thermal properties of an anisotropic medium, i.e. its thermal conductivity coefficients.

If the thermal conductivity tensor of an anisotropic medium is known explicitly and appropriate boundary conditions are specified, then the temperature distribution inside the body may be uniquely determined. A more difficult problem arises if the thermal conductivity coefficients are unknown and have to be determined by supplying some extra information. This may be in the form of overspecified boundary data or interior temperature measurements. This additional information is produced by attaching more sensors onto the boundary of the body or inside it. Thus, in order to reduce the cost of such an experiment it is important to know how many extra sensors are required in order to obtain accurate results. Since the minimum number of sensors required clearly depends on the location of the sensors, an important problem is where to place these sensors such that we obtain reasonable accurate results for the thermal conductivity coefficients with as few extra sensors as possible.

A number of experimental or numerical methods have been proposed to evaluate the thermal conductivity tensor of a heat conductor, or equivalently the hydraulic conductivity tensor in rocks or soils, see [1–4]. It is the purpose of this paper to develop a new boundary element method (BEM), combined with a minimisation technique, to simultaneously predict the unknown conductivity coefficients and the unknown boundary data. The numerical convergence and stability of the method is investigated for various formulations which differ with respect to the type of the extra information provided (heat flux or interior temperature measurements) and to the locations of the sensors which are placed either on

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Nomenclature

$A, B, \ldots$	coefficient matrices
$a, b, c, \ldots$	arbitrary constants
k_{ij}	thermal conductivity coefficients
M	the number of additional measurements
N	number of boundary elements
q_j	given values of the heat flux
R	the radius of the circular domain
R_0	the radius of the circle where interior measurements are taken
s	percentage of noise
T	temperature
T_j	temperature at the nodal point $\underline{\tilde{x}}_j$
T'	heat flux
T'_j	heat flux at the nodal point $\underline{\tilde{x}}_j$
x, y	space variables
$\underline{\tilde{x}}_j$	space nodes

Greek symbols

β	arbitrary constant
ϵ	Gaussian random variable
η	coefficient in the boundary integral equation
Ω	the solution domain
Γ	the boundary of the solution domain
ν	normal vector to a surface
σ	standard deviation

Subscripts and superscripts

j	values at the nodal point
a	the analytical values
n	the numerical values

the boundary of the solution domain or inside the solution domain. A practical way to determine the minimum number of measurements necessary to accurately identify the conductivity coefficients is also suggested. It consists of starting with a small number of measurements and adding additional sensors until all the values of the conductivity coefficients do not change. The convergence and the stability of the method are investigated in detail with respect to increasing the number of extra measurements provided and decreasing the amount of noise added into the input data in order to simulate measurement errors.

2. Mathematical formulation

In this paper, we investigate the steady heat conduction in an anisotropic medium which is in a domain Ω and we assume that heat generation is absent. Therefore the temperature T satisfies the equation

$$k_{11}\frac{\partial^2 T}{\partial x^2}(x,y) + 2k_{12}\frac{\partial^2 T}{\partial x \partial y}(x,y) + k_{22}\frac{\partial^2 T}{\partial y^2}(x,y) = 0 \quad \text{for } (x,y) \in \Omega, \tag{1}$$

where k_{ij} is the thermal conductivity tensor which is assumed to be symmetrical and positive definite so that Eq. (1) is of the elliptic type. Physical and thermal properties of the medium are assumed to be constant and thus the coefficients k_{ij} are independent of the space variables.

If at every point on the surface of the body the temperature, or the heat flux, is known and the temperature is specified at least at one point, and the thermal conductivity tensor is given then the temperature distribution inside the solution domain may be uniquely determined. A different and more difficult situation occurs when the thermal conductivity tensor is unknown and some additional information is supplied. If the temperature is known at all points on the boundary then the extra information that is available may be the heat flux measurements at some points on the boundary, or temperature measurements inside the solution domain.

In this paper, we use a BEM, combined with a least squares technique, to simultaneously provide the unknown boundary data and the thermal conductivity of the medium. In order to illustrate the numerical technique we consider a smooth boundary and for simplicity we take the domain Ω to be $\{(x,y) | x^2 + y^2 < R^2\}$, which is a circle of radius R. We have considered several other smooth boundaries and the general conclusions are similar to those presented here and therefore we have not presented these results.

Following a classical BEM, see [5], the boundary of the solution domain is discretised into a series of small elements Γ_j for $j = \overline{1,N}$ having the end-points $\underline{x}_{j-1}(x_{j-1}, y_{j-1})$ and $\underline{x}_j(x_j, y_j)$ and the mid-point $\underline{\tilde{x}}_j = (\tilde{x}_j, \tilde{y}_j)$. Further, we assume that the elements Γ_j are increasingly numbered in an anti-clockwise direction starting from the point of zero angular polar coordinate.

In this paper, we investigate three formulations which differ with respect to the type of the extra condition imposed (heat flux or temperature measurement) and the location of the sensors attached to the body.

Formulation I. Here we assume that the temperature is given at every point on the boundary $\Gamma = \partial\Omega$ so that the temperature vector

$$\underline{T} = (T_j)_{j=\overline{1,N}} = (T(\underline{\tilde{x}}_j))_{j=\overline{1,N}} \tag{2}$$

is known. More information is given by imposing the heat flux at M consecutive nodal points $(\underline{\tilde{x}}_j, j = 1, M)$,

$$T'_j = \frac{\partial T}{\partial v^+}(\tilde{x}_j) = q_j \quad \text{for } j = \overline{1, M}, \tag{3}$$

where M is a positive integer, v is the normal vector to the surface Γ and

$$\frac{\partial}{\partial v^+} = (k_{11}\cos(v,x) + (k_{21}\cos(v,y))\frac{\partial}{\partial x} + (k_{12}\cos(v,x) + k_{22}\cos(v,y))\frac{\partial}{\partial y}. \tag{4}$$

The convergence of the method is investigated for various numbers of flux measurements, $M \in \{3, \ldots, N\}$.

Formulation II. Here we again assume that the temperature is given at every nodal point but the points where the heat flux are available are spread over the whole boundary Γ. Thus the discretised heat flux boundary condition imposed is given by

$$T'_{(j-1)[N/M]+1} = q_j \quad \text{for } j = \overline{1, M}, \tag{5}$$

where T' and M have the same meaning as in the previous formulation and for every real number x, $[x]$ is the largest integer smaller than x.

Formulation III. Here we assume that the temperature or the heat flux is known at every nodal point and the temperature is also prescribed at M interior points inside the solution domain. For convenience we take the points where these measurements are taken to be equally distributed on a circle of radius $R_0 < R$. Other locations for the extra temperature measurements have been investigated but the results obtained from this investigation best illustrate the ideal location of the extra temperature measurements. The numerical convergence of the method is investigated for various values of the radius R_0.

For all three formulations considered, we use the BEM combined with a least squares technique in order to simultaneously predict the unknown boundary data and the thermal conductivity tensor k_{ij}. The number of heat flux or interior temperature measurements necessary to obtain accurate results is investigated.

It should be noted that the problems which we are considering do not always have a unique solution. This may be illustrated by the following situation. We assume that the temperature T_0 is known at every point on the boundary and inside the solution domain and the heat flux q_0 is known on the whole boundary. Thus, the following equations hold:

$$k_{11}\frac{\partial^2 T}{\partial x^2}(x,y) + 2k_{12}\frac{\partial^2 T}{\partial x \partial y}(x,y) + k_{22}\frac{\partial^2 T}{\partial y^2}(x,y) = 0 \quad (x,y) \in \Omega, \tag{6}$$

$$T(x,y) = T_0(x,y) \quad (x,y) \in \Omega \cup \partial\Omega, \tag{7}$$

$$\frac{\partial T}{\partial v^+}(x,y) = q_0 \quad (x,y) \in \partial\Omega, \tag{8}$$

where T_0 and q_0 are the temperature and the heat flux prescribed. We assume that $\underline{k}^0 = (k^0_{11}, k^0_{12}, k^0_{22})$ is a solution of the problem, i.e., the pair $(\underline{k}^0, T_0)$ satisfies the Eqs. (6)–(8). From Eq. (8) we obtain

$$q_0 = \frac{\partial T_0}{\partial v_0^+} = \nabla T_0 \cdot v_0^+, \tag{9}$$

where v_0^+ is given by the equation

$$v_0^+ = \frac{1}{R}(k^0_{11}x + k^0_{12}y, k^0_{12}x + k^0_{22}y). \tag{10}$$

Next we investigate the uniqueness of the solution of this problem, i.e. we look for a pair $(\underline{k}, T) \neq (\underline{k}^0, T_0)$ which satisfies Eqs. (6)–(8). From Eq. (7) we obtain $T \equiv T_0$ and we look for a vector $\underline{k} \neq \underline{k}^0$ which satisfies

$$k_{11}\frac{\partial^2 T}{\partial x^2} + 2k_{12}\frac{\partial^2 T}{\partial x \partial y} + k_{22}\frac{\partial^2 T}{\partial y^2} = 0 \quad \text{in } \Omega, \tag{11}$$

$$\nabla T_0 \cdot v^+ = \nabla T_0 \cdot v_0^+ \quad \text{on } \partial\Omega, \tag{12}$$

where

$$v^+ = \frac{1}{R}(k_{11}x + k_{12}y, k_{12}x + k_{22}y). \tag{13}$$

Now we illustrate the non-uniqueness of the solution by considering the following three situations:

(i) The temperature is a constant function, namely

$$T_0(x,y) = c \quad \text{for } (x,y) \in \bar{\Omega}. \tag{14}$$

In this case the first- and second-order derivatives of the temperatures are zero and the Eqs. (11) and (12) become trivial. Therefore any vector $\underline{k}$ is a solution of the problem and hence the thermal conductivity tensor of the medium cannot be identified.

(ii) The temperature is a linear function, namely

$$T_0(x,y) = ax + by + c \quad \text{for } (x,y) \in \bar{\Omega}, \tag{15}$$

where a, b and c are constants and $a^2 + b^2 \neq 0$. In this case the second-order derivatives of the temperature are zero so Eq. (11) vanishes and Eq. (12) becomes

$$\frac{a}{R}(k_{11}x + k_{12}y) + \frac{b}{R}(k_{12}x + k_{22}y) = \frac{a}{R}(k^0_{11}x + k^0_{12}y) + \frac{b}{R}(k^0_{12}x + k^0_{22}y) \tag{16}$$

and if we identify the coefficients of x and y we obtain

$$\begin{aligned} ak_{11} + bk_{12} &= ak^0_{11} + bk^0_{12}, \\ ak_{12} + bk_{22} &= ak^0_{12} + bk^0_{22}. \end{aligned} \tag{17}$$

If $a = 0, b \neq 0$ we obtain $\underline{k} = (\beta, k_{12}^0, k_{22}^0), \beta \in \mathbb{R}$ and if $b = 0, a \neq 0$ we obtain $\underline{k} = (k_{11}^0, k_{12}^0, \beta), \beta \in \mathbb{R}$. Hence the solution of the problem is not unique and the thermal conductivity tensor cannot be identified. If $a, b \neq 0$ we obtain

$$\underline{k} = \left(k_{11}^0 - \frac{b}{a}\beta, k_{12}^0 + \beta, k_{22}^0 - \frac{a}{b}\beta\right), \quad \beta \in \mathbb{R}, \tag{18}$$

i.e. the set of solutions of the problem is a one-dimensional subspace of $\mathbb{R}^3$. Again we cannot identify the thermal conductivity tensor of the medium but if one of its components is known then we can find the other two components.

(iii) The temperature is a quadratic function, namely

$$T(x, y) = ax^2 + bxy + cy^2 \quad \text{for } (x, y) \in \bar{\Omega} \tag{19}$$

with non-zero constant coefficients a,b and c. In this case, from Eqs. (11) and (12), we obtain the following system of linear algebraic equations:

$$\begin{aligned} &ak_{11} + bk_{12} + ck_{22} = 0,\\ &2ak_{11} + bk_{12} = 2ak_{11}^0 + bk_{12}^0,\\ &bk_{11} + 2(a + c)k_{12} + bk_{22} = bk_{11}^0 + 2(a + c)k_{12}^0 + bk_{22}^0,\\ &bk_{12} + 2ck_{22} = bk_{12}^0 + 2ck_{22}^0. \end{aligned} \tag{20}$$

Since $\underline{k}_0$ is a solution of the problem we have $ak_{11}^0 + bk_{12}^0 + ck_{22}^0 = 0$ and thus the system (20) may be written as

$$A\underline{k} = A\underline{k}_0, \tag{21}$$

where the matrix A is given by

$$\begin{bmatrix} a & b & c \\ 2a & b & 0 \\ b & 2(a + c) & b \\ 0 & b & 2c \end{bmatrix}. \tag{22}$$

It can be easily seen that if

$$\Delta = (a + c)(b^2 - 4ac) = 0 \tag{23}$$

then $\text{rank}(A) = 2$ and the solution of the system given by Eq. (21) is a one-dimensional subspace of $\mathbb{R}^3$, namely

$$\underline{k} = \left(k_{11}^0 - \frac{b}{2a}\beta, k_{12}^0 + \beta, k_{22}^0 - \frac{b}{2c}\beta\right), \quad \beta \in \mathbb{R}. \tag{24}$$

Thus, we can uniquely identify the thermal conductivity tensor of the medium only if one of its components is known a priori.

It should be noted that the set of solutions given by Eqs. (18) and (24) may be further restricted by imposing the constraints

$$k_{11} \geqslant 0, \quad k_{12} \geqslant 0, \quad k_{22} \geqslant 0, \quad k_{11}k_{22} - k_{12}^2 > 0, \tag{25}$$

which ensures that the thermal conductivity tensor is positive definite, i.e. Eq. (1) is of the elliptic type, but the solution is still not unique. In conclusion, from the three situations considered it may be concluded that if a constant, linear or quadratic temperature is prescribed then the thermal conductivity tensor of the medium cannot be identified, no matter how many extra measurements are taken. Thus, in order to identify the coefficients k_{ij} then more realistic temperatures, such as higher-order polynomials, trigonometric or exponential functions, etc. must be imposed. If this is the situation then the unknown boundary data, as well as the thermal conductivity tensor, may be accurately predicted if some extra information about the heat flux, or the interior temperature, is available. In this paper, we propose a BEM to identify the thermal conductivity tensor and we investigate the numerical convergence with respect to the number of extra measurements provided, the location of the sensors and the level of the noise added into the input data.

3. Description of the method

By applying a classical BEM, see [5], to the governing partial differential equation (1) we obtain the following discretised equation

$$\eta(\underline{x})T(\underline{x}) = \sum_{j=1}^{N} T_j' A_j(\underline{x}, \underline{k}) - \sum_{j=1}^{N} T_j B_j(\underline{x}, \underline{k}), \tag{26}$$

where $\eta(\underline{x}) = 1$ if $\underline{x} \in \Omega$ and $\eta(\underline{x}) = \frac{1}{2}$ if $\underline{x} \in \Gamma$ and the vectors $\underline{T} = (T_j)_{j=1,N}$ and $\underline{T}' = (T_j')_{j=1,N}$ contain the values of the temperature and the heat flux on the boundary. The non-linear functions A_j and B_j depend on the coefficients k_{ij} and may be analytically evaluated, see [6].

Eq. (26), applied at each of the nodal points $\tilde{\underline{x}}_j, j = \overline{1, N}$, gives rise to the following system of N equations

$$\sum_{j=1}^{N} [A_{ij}(\underline{k})T_j' - B_{ij}(\underline{k})T_j] = 0 \quad \text{for } i = \overline{1, N}, \tag{27}$$

where the non-linear functions A_{ij} and B_{ij} are given by

$$A_{ij} = A_j(\tilde{\underline{x}}_i, \underline{k}), \qquad B_{ij} = B_j(\tilde{\underline{x}}_i, \underline{k}) + \frac{1}{2}\delta_{ij}. \tag{28}$$

If the temperature is known at M interior points and Eq. (26) is applied at each of these points then this gives another M equations, namely

$$\sum_{j=1}^{N} [A_{ij}(\underline{k})T_j' - B_{ij}(\underline{k})T_j] = 0 \quad \text{for } i = \overline{N + 1, N + M}. \tag{29}$$

Thus, for the Formulations I and II we obtain a non-linear system of N equations, namely

$$f_i(\underline{k}, \underline{T}, \underline{T}') = 0 \quad \text{for } i = \overline{1, N} \tag{30}$$

with $N - M + 3$ unknowns, while for Formulation III we obtain $N + M$ equations

$$f_i(\underline{k}, \underline{T}, \underline{T}') = 0 \quad \text{for } i = \overline{1, N+M} \tag{31}$$

and $N + 3$ unknowns.

We note that the functions f_i are linear with respect to the vectors $\underline{T}$ and $\underline{T}'$ but highly non-linear with respect to the coefficients $k_{i,j}$. The numerical solution of the problem is constructed by minimising the sum of squares

$$\sum_{i=1}^{K} f_i^2(\underline{k}, \underline{T}, \underline{T}') \to \min \tag{32}$$

under the constraints

$$k_{11} \geqslant 0, \quad k_{12} \geqslant 0, \quad k_{22} \geqslant 0, \tag{33}$$

$$k_{11}k_{22} - k_{12}^2 > 0, \tag{34}$$

where $K = N$ for the Formulations I and II and $K = N + M$ for the Formulation III. Numerically, the sum of the squares (32) is minimised using the NAG subroutine E04UPF, which is designed to minimise an arbitrary smooth sum of squares subject to constraints. This may include simple bounds on the variables, linear constraints and smooth non-linear constraints. The subroutine uses a sequential quadratic programming (SQP) algorithm in which the search direction is the solution of a quadratic programming problem, see [7]. It should be noted that it is important to impose the constraints (33) and (34) in order to generate feasible solutions since unconstrained minimisation was found to produce physically meaningless solutions. In addition, more constraints may be imposed, such as upper bounds on the variables, if any estimates of the maximum values that the conductivity coefficients may take are available. By imposing more constraints then the rate of convergence of the minimisation process increases. Since an initial guess must be specified for the minimisation process, the rate of convergence is also improved if the initial guess is close to any minimum or maximum estimates available for the solution of the problem.

4. Numerical results and discussions

In order to illustrate the numerical technique, the problems formulated are solved in a plane domain $\Omega = \{(x,y) | x^2 + y^2 < 1\}$, i.e. the two-dimensional disc of radius one but other domains, with or without corners, may be considered.

As mentioned in Section 2, in order to uniquely identify the coefficients k_{ij} then the temperature must not be a quadratic, linear or constant function. Therefore, in order to illustrate the convergence of the algorithm we consider the operator L given by

$$L(T)(x,y) = 5\frac{\partial^2 T}{\partial x^2}(x,y) + 4\frac{\partial^2 T}{\partial x \partial y}(x,y) + \frac{\partial^2 T}{\partial y^2}(x,y) = 0 \quad \text{for } (x,y) \in \Omega, \tag{35}$$

which governs the heat diffusion in an anisotropic medium with the thermal conductivity tensor given by $k_{11} = 5, k_{12} = 2$ and $k_{22} = 1$ and the analytical temperature to be retrieved is given by the cubic function

$$T(x,y) = \frac{x^3}{5} - x^2 y + xy^2 + \frac{y^3}{3}. \tag{36}$$

The test example (35) and (36) was chosen such that the determinant of the conductivity coefficients, i.e. $|k_{ij}| = k_{11}k_{22} - k_{12}^2$ is not too small. It is known that the smaller the value of $|k_{ij}|$, the more asymmetric are the temperature fields and the heat flux vectors. Therefore, the smaller the value of $|k_{ij}|$ the more difficult it is to determine the numerical solution. Thus, in order to maintain reasonable accuracy, the determinant of the conductivity coefficients must not be too small, see [8]. Numerous other examples for anisotropic media with different thermal conductivity tensors have been investigated but the same conclusions may be drawn to those using the test example expressed by Eqs. (35) and (36) and therefore the detailed results of these investigations are not presented.

The number of boundary elements used to discretise the boundary of the domain was taken to be $N = 80$ but the results which are obtained for larger values of N are found to have very similar behaviours. The accuracy of the numerical results is investigated for various numbers of measurements, namely $M \in \{3, \ldots, 80\}$.

Arbitrary values may be specified as an initial guess for the unknown boundary data and the conductivity coefficients. The numerical results presented in this paper have been obtained for three different initial guesses for the conductivity coefficients, denoted by (i), (ii) and (iii) as follows:

$$\text{(i)} \quad k_{11} = 1.0, \quad k_{12} = 0.0, \quad k_{22} = 1.0, \tag{37}$$

$$\text{(ii)} \quad k_{11} = 1.0, \quad k_{12} = 0.5, \quad k_{22} = 3.0, \tag{38}$$

$$\text{(iii)} \quad k_{11} = 1.0, \quad k_{12} = 2.0, \quad k_{22} = 4.0, \tag{39}$$

which have been chosen in order to ensure that the initial guess is not too close to the exact solution. A constant zero initial guess was specified for the unknown values of the heat flux or of the temperature on the boundary and these values are not too close to the exact solution.

Starting with these initial guesses, the BEM and a least squares technique are used to simultaneously predict the values of the thermal conductivity tensor and the unknown boundary data.

4.1. Formulation I

In this formulation we apply the method described in Section 3 for three extra heat flux measurements and then we increase the number of extra measurements M by introducing at each step one additional sensor at the next available nodal point. The numerical results for the conductivity coefficients obtained by starting with the initial guess (i) are presented in Fig. 1. It can be seen that for small values of M the solution is inaccurate but if M is larger than about 40 then the exact solution $\underline{k} = (5.0, 2.0, 1.0)$ is retrieved with an error which is less than about 0.25%. It should be mentioned that the number of extra measurements necessary to obtain accurate results depends on the test example and therefore may not be a priori estimated. However, we note that for small values of M the curves obtained for the conductivity coefficients oscillate but then they become constant when M increases and all the coefficients k_{ij} approach their exact values. Therefore in order to obtain an accurate solution we may perform the calculations starting with three extra measurements and increasing the number of measurements M until the numerical results obtained for the conductivity coefficients cease oscillating and become constant. Various other test examples have been investigated and it was found that, in general, the curves obtained for the coefficients k_{ij} have high oscillations for relative low values of M but then they always become horizontal as M increases.

Therefore the method may be applied to identify the conductivity coefficients if we begin with three sensors attached to the body and then add more sensors for heat flux measurements until a constant value for the coefficients k_{ij} is obtained. For example, for the Formulation I and the initial guess (i), the minimum number of measurements for which the numerical solution is accurate is $M = 40$, i.e. the heat flux is required on the whole upper half of the boundary.

Fig. 2 presents the numerical results obtained for the heat flux through the boundary for $M = 20, 30$ and 40 extra measurements. It can be seen that as M increases then the numerical solution approaches the exact solution and for $M = 40$ extra measurements that the numerical solution is a very good approximation to the exact solution. Similar results have been obtained for various initial guesses. Figs. 3–5 present the numerical solutions obtained for each of the conductivity coefficients k_{11}, k_{12} and k_{22}, respectively, for various initial guesses. It can be seen that for all the initial guesses considered, the same algorithm of adding more measurements until the coefficients k_{ij} becomes constant may be applied to identify the number of extra measurements necessary to obtain accurate results. It should be noted that the horizontal parts of the curves obtained by plotting the coefficients k_{ij} as a function of the number of measurements appear at approximately the same value of M for various initial guesses. Therefore, it may be concluded that the minimum number of values of M which are necessary to identify the thermal conductivity tensor depends only on the test example and on the location of the sensors, and it does not depend on the initial guesses for the conductivity coefficients.

Various test examples have been investigated and similar results have been obtained. Therefore it may be concluded that the numerical algorithm presented for

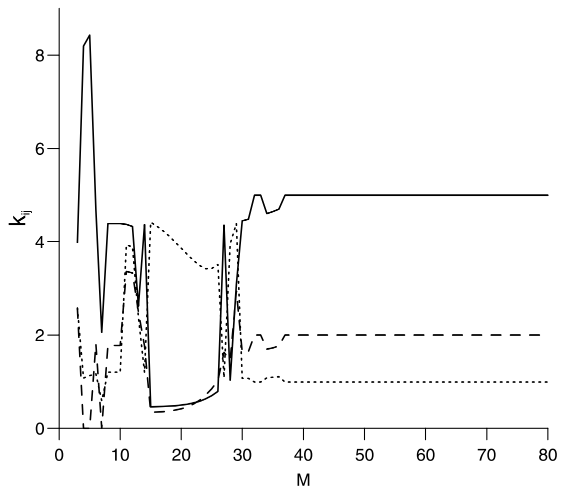

Fig. 1. The numerical solution for the conductivity coefficients k_{11} (–), k_{12} (- - -) and k_{22} ($\cdots$) obtained by starting with the initial guess (i), as a function of the number of heat flux measurements M, for the problem considered in Formulation I.

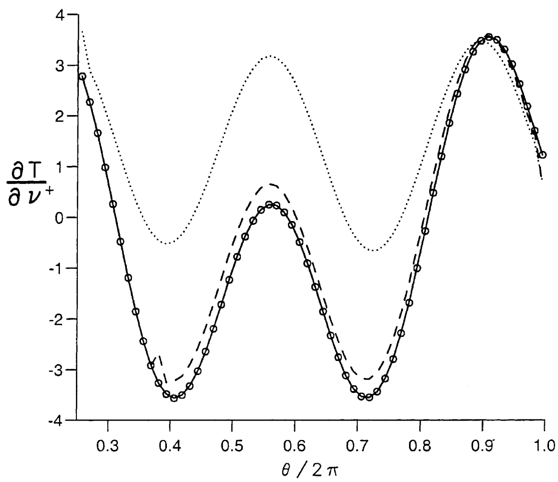

Fig. 2. The numerical solution obtained for the heat flux through the boundary for the inital guess (i) and various numbers of heat flux measurements, namely $M = 20$ ($\cdots$), $M = 30$ (- - -) and $M = 40$ ($\circ$) in comparison with the analytical solution (–), for the problem considered in Formulation I.

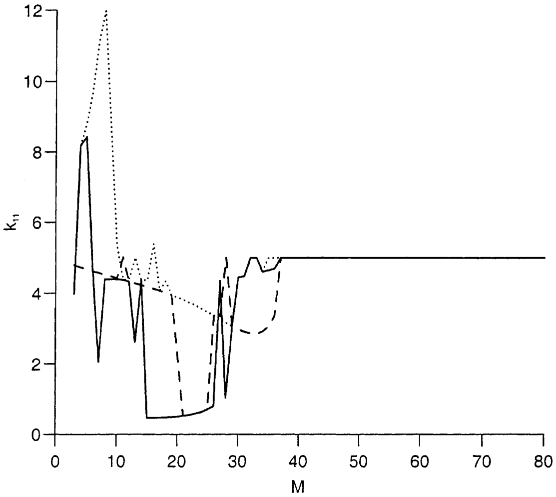

Fig. 3. The numerical solution obtained for the conductivity coefficients k_{11} for various initial guesses for the thermal conductivity tensor, namely, initial guess (i) (–), initial guess (ii) (- - -) and initial guess (iii) (···), for the problem considered in Formulation I.

the Formulation I is convergent with respect to increasing the number of extra measurements.

4.2. Formulation II

In this formulation we use the same method of adding more information until constant values are obtained for the conductivity coefficients. Unlike Formulation I, the heat flux measurements are not taken at consecutive nodal points but instead they are spread over the whole boundary.

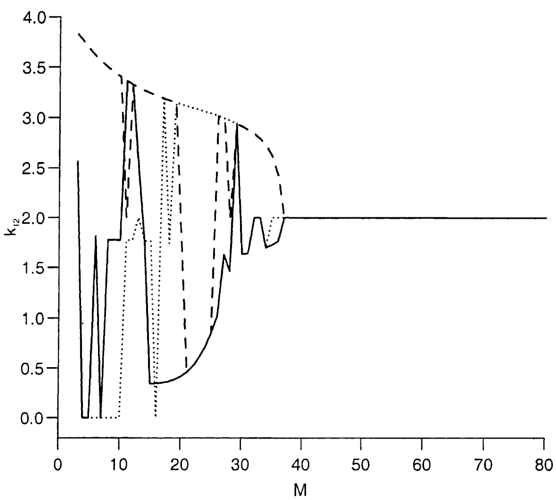

Fig. 4. The numerical solution obtained for the conductivity coefficient k_{12} for various initial guesses for the thermal conductivity tensor, namely, initial guess (i) (–), initial guess (ii) (- - -) and initial guess (iii) (···), for the problem considered in Formulation I.

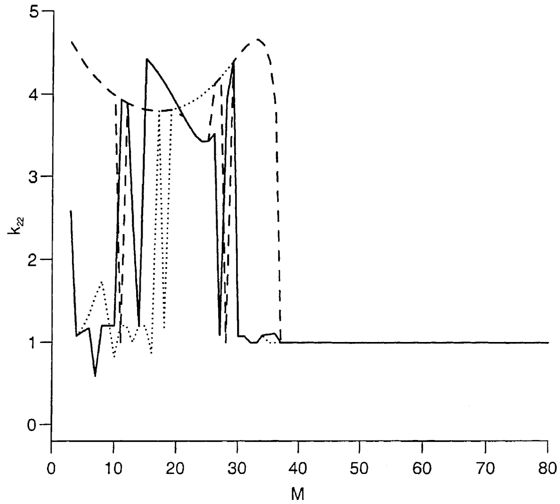

Fig. 5. The numerical solution obtained for the conductivity coefficient k_{22} for various initial guesses for the thermal conductivity tensor, namely, initial guess (i) (–), initial guess (ii) (- - -) and initial guess (iii) (···), for the problem considered in Formulation I.

The numerical results obtained for the conductivity coefficients k_{ij} for the Formulation II, initial guesses (ii) and (iii) and various values of the number of extra measurements are presented in Fig. 6. Even if the nu-

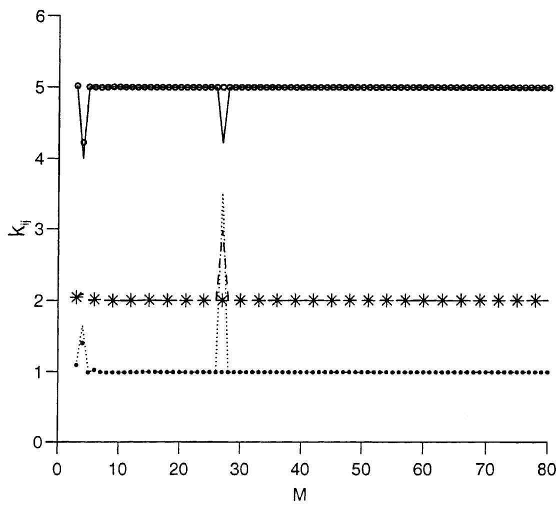

Fig. 6. The numerical solutions obtained for the conductivity coefficients for various initial guesses namely, k_{11} for the guesses (ii) (–), and (iii) (○), k_{12} for the initial guesses (ii) (- - -) and (iii) (*) and k_{22} for the initial guesses (ii) (···) and (iii) (•), for the problem considered in Formulation II.

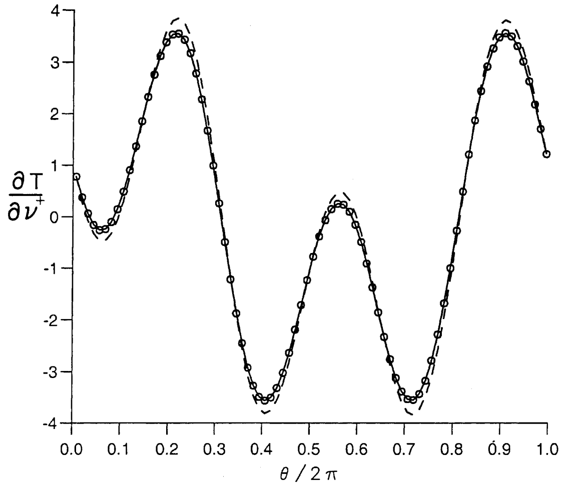

Fig. 7. The numerical solution obtained for the heat flux through the boundary for the initial guess (ii) for $M = 3$ (- - -) and $M = 5$ (∘) heat flux measurements, in comparison with the analytical solution (–) for the problem considered in Formulation II.

merical solutions for the conductivity coefficients are oscillatory for particular values of the M, for example $M = 4$ and $M = 27$ it can be seen that we may identify horizontal parts of the curves k_{ij} even for small values of M. The numerical solution obtained for the heat flux through the boundary for $M = 3$ and 5 heat flux measurements and initial guess (ii) are presented in Fig. 7 and these are found to be in good agreement with the exact solution. Thus, for the test example considered, the conductivity coefficients may be identified using only a small number of heat flux measurements if these measurements are spread over the whole boundary.

Overall, from the numerical results obtained for the Formulations I and II we may conclude that the method presented produces accurate results for the thermal conductivity tensor, provided sufficient heat flux measurements are taken. It was also found that in order to reduce the number of heat flux measurements necessary to obtain accurate results, the measurements should be spread over the whole boundary rather than being grouped on only a part of the boundary. However, if a part of the boundary is not accessible for heat flux measurements, accurate solutions may be obtained if sufficient heat flux measurements are provided on the remainder of the boundary.

4.3. Formulation III

In this last formulation, the temperature is specified at some interior points, rather than specifying extra boundary conditions. We assume that the temperature is known on half of the boundary, namely on $\Gamma_1 = \{z \in \Gamma \,|\, \theta(z) \in [0, \pi)\}$, where θ is the polar angular coordinate. The heat flux is assumed known on the remainder of the boundary, namely on $\Gamma_0 = \{z \in \Gamma \,|\, \theta(z) \in [\pi, 2\pi)\}$. The interior measurements are taken on a circle of radius $R_0 < R$ and the numerical performance of the method is investigated for various values of the radius R_0.

Fig. 8 shows the numerical solution obtained for the conductivity coefficients for various numbers of interior temperature measurements spread over the whole of the circle of radius $R_0 = 0.5$ for the initial guess (i). We note that accurate results are obtained for small numbers of interior temperature measurements. Although not presented here, it is reported that the solution obtained for the unknown boundary data was found to be in very good agreement with the exact solution even for small values of M. Accurate results are obtained for other initial guesses using a small number of extra measurements, as can be seen in Fig. 9 which presents the numerical solutions for the conductivity coefficients obtained using the described algorithm with the initial guesses (ii) and (iii). Similar results are obtained for various values of the radius R_0 where the extra temperature measurements are taken. Fig. 10 presents the error in evaluating the thermal conductivity tensor given by

$$e = \|k_n - k_a\|, \tag{40}$$

where $k_a = (5, 2, 1)$ is the exact solution and k_n is the numerical solution for the conductivity coefficients obtained using the method described, for various values of

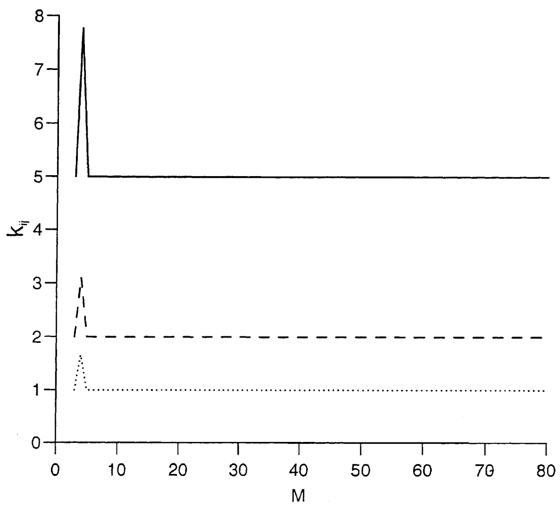

Fig. 8. The numerical solution for the conductivity coefficients k_{11} (–), k_{12} (- - -) and k_{22} (···) obtained by starting with the initial guess (i), as a function of the number of interior temperature measurements M, for the problem considered in Formulation III.

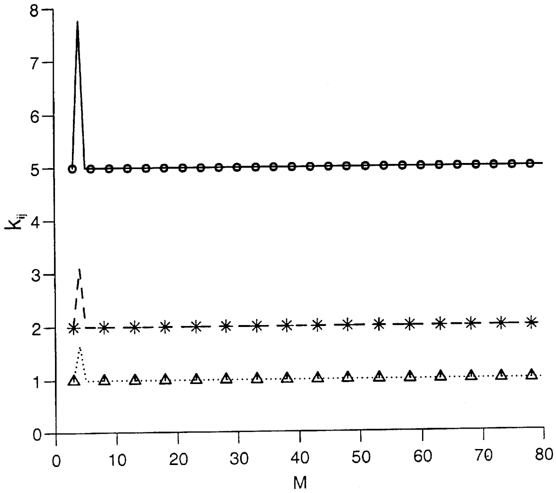

Fig. 9. The numerical solutions obtained for the conductivity coefficients for various initial guesses, namely, k_{11} for the guesses (ii) (–), initial guess (iii) (○), k_{12} for the initial guesses (ii) (- - -) and (iii) (*) and k_{22} for the initial guesses (ii) (···) and (iii) (△), for the problem considered in Formulation III.

the radius R_0 using $M = 40$ extra temperature measurements. It can be seen that the results obtained are comparable for a large interval for the radius R_0 but the accuracy decreases if the measurements are taken too close to the centre of the domain (hence very close to each other) or too close to the boundary (hence too close to the given boundary data). Therefore measurements in these regions should be avoided. This confirms the

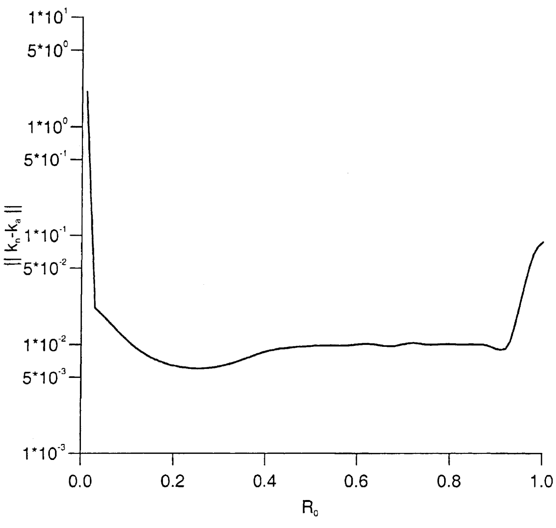

Fig. 10. The error in evaluating the conductivity coefficients for the initial guess (i), as a function of the radius R_0 where the interior temperature measurements are taken, for the problem considered in Formulation III.

conclusion of the previous formulations, namely that the more spread out are the points where the sensors are located, the more accurate are the numerical results.

Overall, from the numerical results obtained for Formulation III it may be concluded that the thermal conductivity tensor may be retrieved with a small number of interior measurements. Therefore, if the heat flux is available only on a small part of the boundary but interior measurements may be taken, then the latter approach should be adopted rather than using many heat flux measurements that are concentrated on a small part of the boundary. However, it should be noted that the results obtained with the heat flux measurements spread over the whole boundary are comparable with those obtained with interior temperature measurements.

5. Stability of the method

Next, the stability of the proposed method is investigated by perturbing the given data with Gaussian random variables of zero mean and standard deviation

$$\sigma = \text{Max}\left(\frac{s}{100}\right), \tag{41}$$

where Max is the maximum value of the input data to be perturbed and s is the percentage of the noise added. Although not presented here, it is reported that, in comparison with their corresponding results which were obtained without noise added into the input data, the curves for k_{ij} obtained with noisy data have only small oscillations but the horizontal parts of the curves are still easily identified. Thus, the same algorithm of identifying the flat parts of the curves k_{ij} may be applied to predict the conductivity coefficients even if the data is noisy.

The numerical results for the conductivity coefficients obtained with $M = 40$ extra measurements for various levels of noise for the three formulations considered are presented in Table 1(a), (b) and (c).

It can be seen that as the amount of noise s decreases then the numerical solution approximates better the exact solution, while remaining stable, for all the formulations considered. Various test examples have been investigated and the same conclusions were drawn. Therefore we may conclude that the proposed numerical algorithm produces a convergent and stable numerical solution with respect to increasing the number of extra measurements and decreasing the amount of noise.

By comparing the results obtained for noisy data for the three formulations considered it can be seen that as the level of noise increases then the error in evaluating the conductivity coefficients increases much faster for the Formulations I and II in comparison with the Formulation III. Therefore it may be concluded that the numerical solution produced by the described algorithm is more sensitive to the noise added into the heat flux

Table 1
The numerical results and the percentage error for the conductivity coefficients obtained with $M = 40$ extra interior temperature measurements and various level of noise added into these measurements, for the problems considered in (a) Formulation I, (b) Formulation II and (c) Formulation III

	k_{11}	k_{12}	k_{22}	Error (%)
(a)				
0% noise	4.996667	1.999085	0.993407	0.135909
1% noise	4.985998	1.998379	1.009449	0.309821
2% noise	4.960864	1.995809	1.047066	1.120178
3% noise	4.934191	1.996782	1.089289	2.025973
(b)				
0% noise	4.996490	1.998825	0.993368	0.138665
1% noise	4.966667	1.985563	0.998294	0.663934
2% noise	4.918598	1.965479	1.019577	1.653405
3% noise	4.858708	1.941663	1.052711	2.952123
(c)				
0% noise	4.992938	1.998069	0.994354	0.168798
1% noise	5.010916	2.003901	0.995275	0.228548
2% noise	5.029023	2.009779	0.996223	0.563391
3% noise	5.047263	2.015703	0.997200	0.910717

measurements than to the noise added into the interior temperature measurements. Thus, if interior temperature measurements are available then they should be used since the numerical solution obtained is more stable than the numerical solution obtained with the same number of heat flux measurements.

6. Conclusions

In this paper, we have proposed a numerical method to identify the thermal properties of a heat conductor using overspecified boundary data or interior temperature measurements for a steady-state heat conduction problem in an anisotropic medium. The numerical algorithm proposed combines a BEM and a least squares technique in order to simultaneously predict the unknown conductivity coefficients and the unknown boundary data. The minimum number of measurements necessary to accurately identify the conductivity coefficients was also investigated. A practical way to identify the thermal properties of the material using the minimum number of sensors was suggested. The method is based on starting with a small number of extra measurements and adding more sensors until constant values are obtained for the thermal conductivity coefficients.

The numerical convergence and stability of the method was investigated for various formulations, different with respect to the type of the extra information provided (heat flux or interior temperature measurements) and to the location of the sensors on the boundary of the solution domain or inside the solution domain. It has been found that the numerical algorithm proposed is convergent with respect to increasing the number of extra measurements. The numerical solution obtained for the thermal conductivity tensor was found to be in good agreement with the exact solution, provided that the number of extra measurements available is sufficiently large.

The identifiability of the inverse problem considered was investigated for both exact and simulated noisy data. Overall, from all three formulations investigated, if may be concluded that the numerical algorithm proposed produces a convergent and stablé numerical solution with respect to increasing the number of extra measurements provided and decreasing the amount of noise added into the input data. Thus, the method proposed provides an accurate means of recovering the material properties.

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